Lecture 8.

Let $F:\mathbb{R}\to\mathbb{R}$ be increasing + right cont. and μ_F the complete Lebesgue-Stieltjes measure.

$\mathcal{M}_{\mu_F}$ denotes the σ-algebra (containing $\mathcal{B}_{\mathbb{R}}$) of μ_F. Recall that, by construction, for $E\in\mathcal{M}_{\mu_F}$,

$$\mu_F(E)=\inf\Big\{\sum_{k=1}^{\infty}\underbrace{F(b_k)-F(a_k)}_{\mu((a_k,b_k])},\quad E\subseteq\bigcup_{k=1}^{\infty}(a_k,b_k]\Big\}$$

Note that using half-open intervals was convenient for the construction of μ_F, but they are special to $\mathbb{R}$ so would be useful to replace them at this point by open intervals.

In fact, we have

<u>Thm!</u>. For all $E \in \mathcal{M}_{\mu_F}$,

$$\mu_F(E) \underset{(1)}{=} \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}$$

$$\underset{(2)}{=} \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

<u>Pf.</u> (1). First, note that this is trivial if $\mu_F(E)=\infty$. Pick $\varepsilon>0$ and cover E by $\bigcup_{n=1}^{\infty}(a_n,b_n]$ s.t. $\mu_F(E) \geq \sum_{n=1}^{\infty}(F(b_n)-F(a_n))-\varepsilon/2$

Next, use right cont. to find $b_n' > b_n$ s.t. $F(b_n) > F(b_n') - \varepsilon \cdot 2^{-k}$. Then

$E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n') =: U$ open, $\mu_F(\bigcup_{n=1}^{\infty}(a_n,b_n')) \leq$

$$\leq \sum_{n=1}^{\infty}\underbrace{(F(b_n')-F(a_n))}_{\mu_F((a_n,b_n'])} \leq \sum_{n=1}^{\infty}(F(b_n)-F(a_n)) + \varepsilon/2$$

$$\leq \mu_F(E) + \varepsilon/2 + \varepsilon/2 \Rightarrow \mu_F(U) \leq \mu_F(E) + \varepsilon.$$

Since $\mu_F(E) \le \mu_F(U)$ for all $U \supseteq E$, we conclude

$$\mu_F(E) = \inf\{\mu_F(U) : E \subseteq U \text{ open}\}.$$

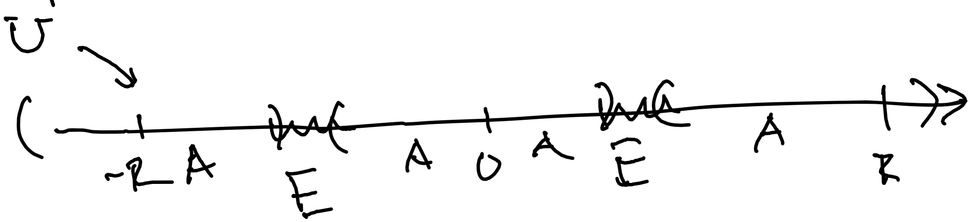
(2) Let $E_n = E \cap [-n,n]$. Then, each E_n is bdd, $E_1 \subseteq E_2 \subseteq \dots$, $E = \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow$ $\mu_F(E) = \lim_{n\to\infty} \mu_F(E_n)$

cont. from below

If we can show $\mu_F(E_n) = \sup\{\mu_F(K) : E \supseteq K \text{ compact}\}$

then $\mu_F(E) = \sup\{\mu_F(K) : \dots\}$ follows easily. Thus, <u>WLOG</u> assume $E \subseteq \mathbb{R}$ is <u>bdd</u>, say $E \subseteq [-R,R]$. Let $A = [-R,R] \setminus E$ and, given $\varepsilon > 0$, take U open s.t. $A \subseteq U$ and $\mu_F(A) \ge \mu_F(U) - \varepsilon$



U^c is closed and $U^c \subseteq A^c = E \cup [-R,R]^c$

Since $E \subseteq [-R,R]$, $K = U^c \cap [-R,R] \subseteq E$ and K is closed + bdd $\Rightarrow$ compact.

$\mu_F([-R,R]) \leq \mu_F(U) + \mu_F(K)$

$\mu_F(E) + \mu_F(A) = \mu_F([-R,R]) \leq \mu_F(U) + \mu_F(K)$

$\leq \mu_F(A) + \varepsilon + \mu_F(K) \Rightarrow$

$\mu_F(E) \leq \mu_F(K) + \varepsilon \Rightarrow \mu_F(E) = \sup\{\mu_F(K) : E \supseteq K \text{ compact}\}$.

$\boxtimes$

This "regularity" of the Lebesgue-Stieltjes leads to a description of the sets in $\mathcal{M}_{\mu_F}$ as follows.